

The diophantine equation $\frac{x^3}{3} + y^3 + z^3 - 2xyz = 0$

Joseph Amal Nathan

Reactor Physics Design Division, Bhabha Atomic Research Centre, Mumbai-400085, India
email:josephan@magnum.barc.ernet.in

Abstract: We will be presenting two theorems in this paper. The first theorem, which is a new result, is about the non-existence of integer solutions of the cubic diophantine equation. In the proof of this theorem we have used some known results from theory of binary cubic forms and the method of infinite descent, which are well understood in the purview of Elementary Number Theory(ENT). In the second theorem, we show, that the famous Fermat's Last Theorem(FLT) for exponent 3 and the first theorem are equivalent. So Theorem1 and 2 constitute an alternate proof for the non-existence of integer solutions of this famous cubic Fermat's equation. It is well known, that from L.Euler(1770) to F.J.Duarte(1944) many had given proof of FLT for exponent 3. But all proofs uses concepts, which are beyond the scope of ENT. Hence unlike other proofs the proof given here is as an ENT proof of FLT for exponent 3.

We will denote the greatest common divisor of integers a_1, a_2, a_3, \dots by symbol (a_1, a_2, a_3, \dots) . For the results from theory of Binary Cubic Forms, we follow L.J.Mordell[1]. Consider the binary cubic

$$f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 = \{a, b, c, d\},$$

with integer coefficients and discriminant

$$D = -27a^2d^2 + 18abcd + b^2c^2 - 4ac^3 - 4b^3d,$$

where $D \neq 0$. The quadratic covariant $H(x, y)$ is

$$\begin{aligned} H(x, y) &= (b^2 - 3ac)x^2 + (bc - 9ad)xy + (c^2 - 3bd)y^2, \\ &= Ax^2 + Bxy + Cy^2 = \{A, B, C\} \end{aligned} \quad (1)$$

with discriminant $B^2 - 4AC = -3D$ and the cubic covariant

$$\begin{aligned} G(x, y) &= -(27a^2d - 9abc + 2b^3)x^3 + 3(6ac^2 + b^2c - 9abd)x^2y \\ &\quad + 3(bc^2 - 6b^2d + 9acd)xy^2 + (27ad^2 - 9bcd + 2c^3)y^3. \end{aligned}$$

Lemma1: $f(x, y), H(x, y)$ and $G(x, y)$ are algebraically related by the identity,

$$G^2(x, y) + 27Df^2(x, y) = 4H^3(x, y) \quad [1].$$

Lemma2: All integer solutions of

$$X^2 + 27kY^2 = 4Z^3, \quad (X, Z) = 1 \quad (2)$$

are given by taking, $X = G(x, y)$, $k = D$, $Y = f(x, y)$, $Z = H(x, y)$.

Proof. For proof of this lemma, we follow word by word the proof given in L.J.Mordell[1] for equation of the form $X^2 + kY^2 = Z^3$.

Let $[X, Y, Z] = [g, f, h]$ be a solution such that

$$g^2 + 27kf^2 = 4h^3, \quad (g, h) = 1. \quad (3)$$

We will construct a binary cubic $f(x, y)$ with integer coefficients and of discriminant $D = k$ such that g, f, h are values assumed by $G(x, y), f(x, y), H(x, y)$ for integer x, y . Then all solutions of (2) are given by taking $f(x, y)$ a set of binary cubics of discriminant D and letting x, y run through all integer values for which $(X, Z) = 1$.

Since from (3), $-3k$ is a quadratic residue of h , there exist binary quadratics of discriminant $-3D$ with first coefficient h ,

$$\{h, B, C\} = hx^2 + Bxy + Cy^2, \quad \text{where } B^2 - 4hc = -3k.$$

We take for B any solution of the congruence $3fB \equiv -g \pmod{4h^3}$. We will now construct a binary cubic $\{f, b, c, d\}$ with discriminant $D = k$ and $H(x, y)$ given in (1) by $\{h, B, C\}$. Since,

$$h = b^2 - 3fc, \quad c = \frac{b^2 - h}{3f},$$

we can take $b \equiv g/2h \pmod{f}$ and in particular, $2bh = g + 3fB$, and so $b \equiv 0 \pmod{2h^2}$. Then

$$\begin{aligned} 4h^2c &= \frac{(g + 3fB)^2 - 4h^3}{3f} \\ &= -9kf + 2gB + 3fb^2. \end{aligned}$$

We now show c is an integer,

$$\begin{aligned} 3f \cdot 4h^2c &\equiv 3f \left[-9kf + 2g \left(-\frac{g}{3f} \right) + 3f \left(\frac{g}{3f} \right)^2 \right] \pmod{h^2} \\ &\equiv -27kf^2 - g^2 \pmod{h^2} \equiv 0 \pmod{h^2}. \end{aligned}$$

We now find d . Since $bc - 9fd = B$,

$$9fd = \left(\frac{g + 3fB}{2h} \right) \left(\frac{-9kf + 2gB + 3fb^2}{4h^2} \right) - B.$$

We will simplify the above equation and show d is an integer for completion of proof.

$$24h^3d = -3kg - 27kfB + 3gB^2 + 3fB^3,$$

$$\begin{aligned} 9f^2 \cdot 24h^3d &\equiv 9f^2 \left[-3kg + 9kg + \frac{3g^3}{9f^2} - \frac{g^3}{9f^2} \right] \pmod{h^3} \\ &\equiv 2g [27kf^2 + g^2] \pmod{h^3} \equiv 0 \pmod{h^3}. \end{aligned} \quad \square$$

Lemma3: For $D = 1$ the classes of binary cubics are given by $a = 0$, $|b| = 1$, $|c| = 1$, $d = 0$.

Proof. The discriminant D is also the invariant

$$D = a^4(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2,$$

where α , β , γ are the roots of the equation

$$a\zeta^3 + b\zeta + c\zeta + d = 0.$$

When $D > 0$ the roots α , β , γ are all real and distinct. So from (1) for expressions of A we get

$$\frac{A}{a^2} = \left(\frac{b}{a}\right)^2 - 3\left(\frac{c}{a}\right) = (\alpha^2 + \beta^2 + \gamma^2) - (\alpha\beta + \beta\gamma + \gamma\alpha) > 0$$

Since D , $A > 0$, if $H(x, y) = \frac{1}{4A} [(2Ax + By)^2 + 3Dy^2] = 0 \Rightarrow x = y = 0$. Hence for $D > 0$ we have, $B^2 - 4AC < 0$ and $A > 0$. Then $H(x, y)$ is positive definite. We know a binary quadratic form $\{I, J, K\}$ is reduced if $K \geq I \geq |J|$. Since every class of positive definite binary forms contains at least one reduced form, we can take $C \geq A \geq |B|$. So $AC \geq B^2$, from $4AC - B^2 = 3D$, we get $AC \leq D$ and $A \leq \sqrt{D}$.

Now for $D = 1$ the above inequalities give, $A \leq 1$, $C \leq 1$ and $A, C \geq |B| \Rightarrow |B| \leq 1$. Since $A, C > 0$, $A = C = 1$. From $4AC - B^2 = 3D$ we get $B^2 = 1 \Rightarrow |B| = |bc - 9ad| = 1$, so $|b|, |c| \geq 1$, since none of b, c can be zero. Substituting for a, d from $b^2 - 3ac = 1$, $c^2 - 3bd = 1$, in $|B| = 1$ we get,

$$b^2 + c^2 - 1 = \pm bc.$$

Let $|b| \geq |c| > 1$. From above equation $b^2 + c^2 - 1 \leq |bc| \Rightarrow b^2 + c^2 - 1 \leq b^2 \Rightarrow c^2 \leq 1$ contradicting $|c| > 1$. So $|b| = |c| = 1$ and $a = d = 0$. \square

Later in this paper, we will require (2) with $k = 1$. From Lemma3 for $D = 1$ we have $|b| = |c| = 1$ and $a = d = 0$. Since $k = D = 1$ from the form of (2), we see, it is sufficient to take $a = 0$, $b = \pm 1$, $c = 1$, $d = 0$. So we have,

$$f(x, y) = xy(x \pm y), \quad H(x, y) = x^2 \pm xy + y^2, \quad G(x, y) = 2x^3 \pm 3x^2y - 3xy^2 \mp 2y^3. \quad (4)$$

Lemma4: Given nonzero integers x, y such that $(x, y) = 1$ for any positive odd integer n ,

$$\left(x + y, \frac{x^n + y^n}{x + y}\right) = (x + y, n).$$

Proof. Let $(x + y, n) = g$. Since, $\frac{x^n + y^n}{g(x + y)} = \frac{(x + y)}{g} \left[\sum_{i=1}^{n-1} (-1)^{i-1} i x^{n-i-1} y^{i-1} \right] + \frac{n}{g} y^{n-1}$, any divisor of $\frac{(x + y)}{g}$ will divide all terms except last. Hence $\left(\frac{x + y}{g}, \frac{x^n + y^n}{g(x + y)}\right) = 1$. \square

Theorem1: *The cubic equation*

$$\frac{u^3}{3} + v^3 + w^3 - 2uvw = 0, \quad (5)$$

has no solution in nonzero integers u, v, w .

Proof: There is no loss of generality if u, v, w are pairwise prime integers. From (5) we see $3 \mid u$ and $u \mid (v^3 + w^3) \Rightarrow 9 \mid (v^3 + w^3) \Rightarrow 9 \mid u$.

We can always choose nonzero integers M, N, p, q such that,

$$v + w = 3Mp, \quad v^2 - vw + w^2 = 9M^2p^2 - 3vw = 3Nq, \quad u = 9pq,$$

where $3 \nmid M$ and $3 \mid p$ if and only if $3^3 \mid u$. One can see N, q are odd integers. By Lemma4 we have $3 \parallel (v^2 - vw + w^2)$ hence,

$$3 \nmid N, q \quad \text{and} \quad (M, N) = (M, q) = (p, N) = (p, q) = 1. \quad (6)$$

Substituting for $u, v + w, vw$ in terms of M, N, p, q in (5) and after rearranging we get

$$3p^2(9q^2 - 2M^2) = -N(M + 2q). \quad (7)$$

Now using (7) we will find M, N in terms of p, q . Since q is odd $9q^2 - 2M^2$ is odd. Let $(9q^2 - 2M^2, M + 2q) = \delta$. Since $9q^2 - 2M^2 = q^2 + 2(4q^2 - M^2)$ any prime divisor ϵ of δ divides $q \Rightarrow \epsilon \mid M$. But $(M, q) = 1$ hence $\delta = 1$. Since $(N, 3p^2) = 1$ from (7) we have,

$$M = 3p^2 - 2q \quad \text{and} \quad N = 2M^2 - 9q^2 = 18p^4 - 24p^2q - q^2.$$

We can see $u, v + w, vw$ are integers. If $v - w$ is also an integer it shows the existence of integer solutions for (5) in terms of integers p, q . Substituting for $v + w, vw$ in the following expression we get,

$$(v - w)^2 = (v + w)^2 - 4vw = -4q^3 - 27[p(p^2 - 2q)]^2,$$

which can be written as,

$$(v - w)^2 + 27[p(p^2 + 2(-q))]^2 = 4(-q)^3.$$

From Lemma2 all integer solutions of the above equation is given by taking,

$$X = G(x, y) = v - w, \quad k = D = 1, \quad Y = f(x, y) = p(p^2 - 2q), \quad Z = H(x, y) = -q.$$

Since $k = D = 1$, substituting for $f(x, y), H(x, y)$ from (4) in above equation for Y ,

$$p[p^2 + 2(x^2 \pm xy + y^2)] = xy(x \pm y) \quad (8)$$

Let $(x, y) = \delta$. $q = -(x^2 \pm xy + y^2)$ so any prime divisor ϵ of δ divides q . From (8) we see ϵ divides either p or $p^2 - 2q \Rightarrow \epsilon \mid p$. But $(p, q) = 1$ hence $\delta = 1$.

To find the integer solutions of (8), we follow a similar procedure as we did for (5). It is possible to choose nonzero integers Q, R, S, T such that

$$QR = xy, \quad ST = x \pm y, \quad p = RT.$$

Since $(x, y) = 1$, we have $(Q, S) = (Q, T) = (R, S) = (R, T) = 1$. Substituting for x, y in terms of Q, R, S, T in (8) and after simplifying we get,

$$T^2(R^2 + 2S^2) = Q(S \pm 2R). \quad (9)$$

Now we will show $3 \mid f(x, y)$. From (6), we have $3 \nmid q$. Assume $3 \nmid f(x, y)$ then $x \equiv \pm y \pmod{3} \Rightarrow q \equiv 0 \pmod{3}$ a contradiction. Now let $(Q, R) = \delta$ any prime divisor ϵ of δ from (9) will divide $(R^2 + 2S^2)$ since $(Q, T) = 1$. Hence ϵ has to divide 2 or S . Since $(Q, S) = 1$ we have $\epsilon \leq 2 \Rightarrow (Q, R) \leq 2$. Similarly let $(S, T) = \delta$ any prime divisor ϵ of δ again from (9) will divide $(S \pm 2R)$ since $(Q, T) = 1$. So ϵ has to divide 2 or R . Since $(R, T) = 1$ we get $(S, T) \leq 2$. So we see $3 \mid Q$ or R or S or T .

To evaluate Q, S in terms of R, T using (9) we should know about $(R^2 + 2S^2, S \pm 2R)$. Let $(R^2 + 2S^2, S \pm 2R) = \Delta$ and η be a prime divisor of Δ . Since $R^2 + 2S^2 = 2(S - 2R)(S + 2R) + 9R^2$, $\eta \mid 3$ or R . If $\eta \mid R$ then $\eta \mid S \Rightarrow \Delta = 1$ since $(R, S) = 1$. If $\eta \mid 3$ then Δ can only be a power of 3.

Case-I: For following cases $\Delta = 1$.

(i) $3 \mid Q, (S \pm 2R)$. From (9) we choose the case $T^2[2(S - 2R)(S + 2R) + 9R^2] = Q(S \mp 2R)$. Since $(R, S) = 1$ we have $(2R, S) \leq 2$ and so $3 \nmid (S \mp 2R)$. Any prime divisor η of Δ divides R and from above argument we get $\Delta = 1$. The other part $T^2(R^2 + 2S^2) = Q(S \pm 2R)$ when $3 \mid Q, (S \pm 2R)$ will be taken in Case-II.

(ii) $3 \mid R$ or S . In both the cases $3 \nmid (S \pm 2R)$. Since $R^2 + 2S^2 = 2(S^2 - 4R^2) + 9R^2$ any prime divisor of Δ divides R so $\Delta = 1$.

For (i) and (ii) we have $(R^2 + 2S^2, S \pm 2R) = 1$ and $(Q, T) = 1$. So from (9)

$$S = T^2 \mp 2R, \quad Q = R^2 + 2S^2 = 9R^2 + 2T^4 \mp 8T^2R.$$

As we did before when we evaluate $x \mp y$ in terms of R, T we get,

$$(x \mp y)^2 = (x \pm y)^2 \mp 4xy = \mp 36R^3 + T^2(T^2 \mp 6R)^2. \quad (10)$$

We have $(R, T) = 1$ and $3 \nmid T$. Let $(36R^3, T^2(T^2 \mp 6R)^2) = \delta$. Let R be even or odd and T be odd then $T(T^2 \mp 6PL)$ is odd. From (10) $(x \mp y)$ is odd and $2 \nmid \delta$, then any prime divisor of δ has to divide $T \Rightarrow \delta = 1$. If R is odd and T is even, $4 \mid T(T^2 \mp 6R)$. From (10) $2 \parallel (x \mp y)$ and $\delta = 4$. Since $\delta = 1$ or $4 \Rightarrow ((x \mp y)^2, T^2(T^2 \mp 6R)^2) = 1$ or 4 respectively we get $((x \mp y) + T(T^2 \mp 6PL), (x \mp y) - T(T^2 \mp 6PL)) = 2$. We will see there will be no loss of generality if we take $(x \mp y) - T(T^2 \mp 6R)$ is divisible by 3. Now from (10) we get the following equations,

$$(x - y) + T(T^2 - 6PL) = \pm 2P^3 \quad \text{and} \quad (x - y) - T(T^2 - 6PL) = \mp 18L^3, \quad (11)$$

$$(x + y) + T(T^2 + 6PL) = +2P^3 \quad \text{and} \quad (x + y) - T(T^2 + 6PL) = +18L^3, \quad (12)$$

where $R = PL$ and $(P, L) = 1$. Eliminating $(x - y)$ from (11) and $(x + y)$ from (12),

$$T(T^2 - 6PL) = \pm P^3 \pm 9L^3, \quad T(T^2 + 6PL) = P^3 - 9L^3.$$

Substituting for $\{3L = \mp r, P = \mp s, T = t\}$ and $\{3L = r, P = -s, T = t\}$ in the above first and second equations respectively we get,

$$\frac{r^3}{3} + s^3 + t^3 - 2rst = 0,$$

which has the form of (5). Since we have $u = 9pq$, $p = RT$, $R = PL$, $|3L| = |r|$, we see

$$|u| > |3p| \geq |3R| \geq |3L| = |r|.$$

Hence by the method of infinite descent, there will be no solution in nonzero integers u, v, w for (5).

Now we take the remaining cases for the completion of the proof.

Case-II: For remaining cases $\Delta = 3^2$.

(i) $3 \mid Q, (S \pm 2R)$. We have to take $T^2[2(S - 2R)(S + 2R) + 9R^2] = Q(S \pm 2R)$. We have $3 \nmid (S \mp 2R)$ and $3^2 \mid (S \pm 2R)$. If $3^3 \mid (S \pm 2R)$ then $3^4 \mid Q(S \pm 2R)$ but $3^2 \nmid (R^2 + 2S^2) \Rightarrow 3 \nmid T$ a contradiction. So $3^2 \parallel (S \pm 2R)$ which gives $\Delta = 3^2$.

(ii) $3 \mid T$. Since $(Q, T) = 1$ we see from (9) $3^2 \mid (S \pm 2R), (R^2 + 2S^2)$ and again from (9) we get $3^4 \mid (S \pm 2R) \Rightarrow 3^2 \parallel (R^2 + 2S^2) \Rightarrow \Delta = 3^2$

For (i) and (ii) we have $(R^2 + 2S^2, S \pm 2R) = 3^2$ and $(Q, T) = 1$. Hence from (9) we get

$$S = 9T^2 \mp 2R, \quad 9Q = R^2 + 2S^2 = R^2 + 18T^4 \mp 8T^2R.$$

Evaluating $x \mp y$ in terms of R, T we get,

$$(x \mp y)^2 = \mp 4R^3 + T^2(9T^2 \mp 6R)^2. \quad (13)$$

We have $(R, T) = 1$ and $3 \nmid R$. Let $(4R^3, T^2(9T^2 \mp 6R)^2) = \delta$. Let R be even or odd and T be odd then $T(9T^2 \mp 6PL)$ is odd. From (13) $(x \mp y)$ is odd and $2 \nmid \delta$, then any prime divisor of δ has to divide $T \Rightarrow \delta = 1$. If R is odd and T is even, $4 \mid T(9T^2 \mp 6R)$. From (13) $2 \parallel (x \mp y)$ and $\delta = 4$. Since $\delta = 1$ or $4 \Rightarrow ((x \mp y)^2, T^2(9T^2 \mp 6R)^2) = 1$ or 4 respectively we get $((x \mp y) + T(9T^2 \mp 6PL), (x \mp y) - T(9T^2 \mp 6PL)) = 2$. From (13),

$$(x - y) + T(9T^2 - 6PL) = \pm 2P^3 \quad \text{and} \quad (x - y) - T(9T^2 - 6PL) = \mp 2L^3, \quad (14)$$

$$(x + y) + T(9T^2 + 6PL) = +2P^3 \quad \text{and} \quad (x + y) - T(9T^2 + 6PL) = +2L^3, \quad (15)$$

where $R = PL$ and $(P, L) = 1$. As before eliminating $(x - y)$ from (14) and $(x + y)$ from (15),

$$T(9T^2 - 6PL) = \pm P^3 \pm L^3, \quad T(9T^2 + 6PL) = P^3 - L^3.$$

Substituting for $\{3T = r, P = \mp s, L = \mp t\}$ and $\{3T = r, P = -s, L = t\}$ in above first and second equations respectively we again arrive at,

$$\frac{r^3}{3} + s^3 + t^3 - 2rst = 0,$$

We have $u = 9pq$, $p = RT$, $|3T| = |r|$, so

$$|u| > |3p| \geq |3T| = |r|.$$

Again there are no solution in nonzero integers u, v, w for (5). □

R. Perrin[2] had shown the following fact concerning FLT for exponent 3,

The following statements are equivalent and true:

- (1) *Fermat's last theorem is true for the exponent 3.*
- (2) *For every $n \geq 1$ the equation*

$$X^3 + Y^3 + 3^{3n-1}Z^3 = 2 \cdot 3^n XYZ$$

has no solution in nonzero integers X, Y, Z , not multiples of 3.

We will show that the above equation and (5) are same. In (5) we have $3 \mid u$ and $3 \nmid v, w$. Let $n \geq 1$ such that $3^n \parallel u$. If we take $u = 3^n Z$, $v = X$, $w = Y$ we see $3 \nmid XYZ$. After substituting for u, v, w in (5) we get the above equation.

Now we state the following theorem wherein we show Theorem1 and FLT for exponent 3 are equivalent by an independent method.

Theorem2: *The equation $x^3 + y^3 + z^3 = 0$ has no solution in nonzero integers x, y, z .*

Proof: There is no loss in generality if x, y, z are pairwise prime. Define $m = x + y + z$, we see $3 \mid m$. From $[m - (x + y)]^3 = z^3$ we get

$$m^3 - 3m^2(x + y) + 3m(x + y)^2 - 3xy(x + y) = 0 \Rightarrow 3 \mid xyz.$$

Let $3 \mid z$. Now $-z^3 = (x + y)(x^2 - xy + y^2)$, from Lemma4,

$$\begin{array}{lll} 3(x + y) = u^3, & x^2 - xy + y^2 = 3U^3, & z = -uU \\ (z + x) = v^3, & z^2 - zx + x^2 = V^3, & y = -vV \\ (z + y) = w^3, & y^2 - zy + z^2 = W^3, & x = -wW \end{array}$$

where u, v, w, U, V, W are relatively prime nonzero integers and $3 \mid u$. From the above equations we get,

$$2z = -\frac{u^3}{3} + v^3 + w^3, \quad 2y = \frac{u^3}{3} - v^3 + w^3, \quad 2x = \frac{u^3}{3} + v^3 - w^3, \quad 2m = \frac{u^3}{3} + v^3 + w^3. \quad (16)$$

Also from the definition of m we get

$$m = u \left(\frac{u^2}{3} - U \right) = v(v^2 - V) = w(w^2 - W) \Rightarrow m = \Gamma uvw, \quad (17)$$

where $\Gamma = \left(\frac{u^2}{3} - U, v^2 - V, w^2 - W \right)$. To evaluate Γ , after substituting for z , y , x , m , from (16), in the following expression, we get,

$$\Gamma = \sqrt[3]{\frac{x^3 + y^3 + z^3 + m^3}{u^3 v^3 w^3}} = 1.$$

Equating the expressions of m in (16) and (17) we get,

$$\frac{u^3}{3} + v^3 + w^3 - 2uvw = 0,$$

where u , v , w are pairwise relatively prime integers. Now from Theorem 1 there are no nonzero integers u , v , w satisfying the above equation. Hence the equation $x^3 + y^3 + z^3 = 0$ has only trivial solutions in integers. \square

Acknowledgement. I thank M.A. Prasad for many fruitful discussions and valuable suggestions.

REFERENCES

1. L.J. Mordell *Diophantine Equations*, ACADEMIC PRESS London and New York, 1969, Chapter 24.
2. Paulo Ribenboim *Fermat's Last Theorem for Amateurs*, Springer, 1999, Chapter VIII.